Sect 9.4 - Properties of Triangles

Objective a: Understanding congruent triangles.

Two triangles are **congruent** if they have the same shape and are the same size. The notation for writing that triangle ABC is congruent to triangle HET is **$\Delta ABC \cong \Delta HET$**. The ordering of the letters shows the corresponding vertices. In this case, $\angle A$ corresponds to $\angle H$, $\angle B$ corresponds to $\angle E$ and $\angle C$ corresponds to $\angle T$. Corresponding sides and corresponding angles of triangles are called corresponding parts. In congruent triangles, the corresponding parts always have the same measure. If $\Delta ABC \cong \Delta HET$, then

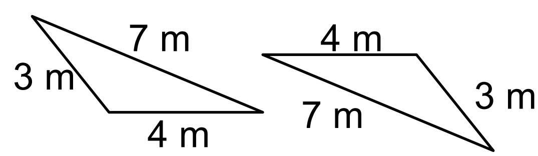
$m\angle A = m\angle H$	$\overline{AB} \cong \overline{HE}$ or $AB = HE$
$m\angle B = m\angle E$	$\overline{BC} \cong \overline{ET}$ or $BC = ET$
$m\angle C = m\angle T$	$\overline{AC} \cong \overline{HT}$ or $AC = HT$

In examining two triangles, the question occurs as to when there is enough information given to determine for sure that two triangles are congruent. To answer that question, we will use the following properties:

SSS Property

If three sides of one triangle are congruent to the corresponding three sides of another triangle, then the triangles are congruent.

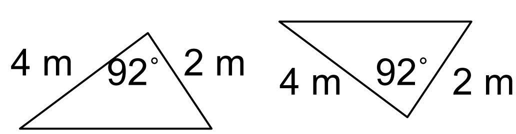
The triangles on the right are congruent by the SSS property.



SAS Property

If two sides and the angle between them in one triangle are congruent to two sides and the angle between them in another triangle, then the triangles are congruent.

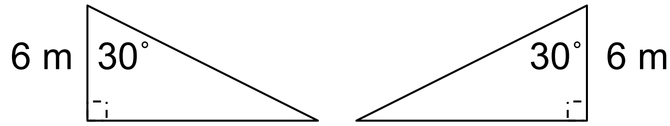
The triangles on the right are congruent by the SAS property.



ASA Property

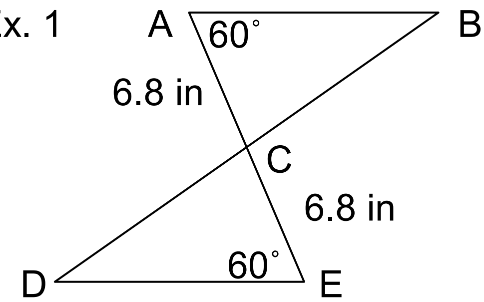
If two angles and the side between them in one triangle are congruent to two angles and the side between them in another triangle, then the triangles are congruent.

The triangles on the right are congruent by the ASA property.



Determine if the following triangles are congruent:

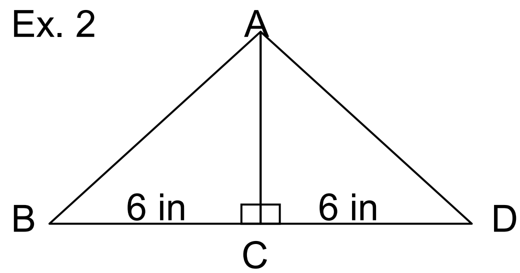
Ex. 1



Solution:

Since ∠DCE ≅ ∠BCA, then ΔDCE ≅ ΔBCA by the ASA property.

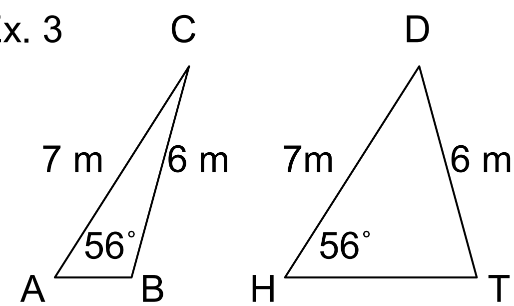
Ex. 2



Solution:

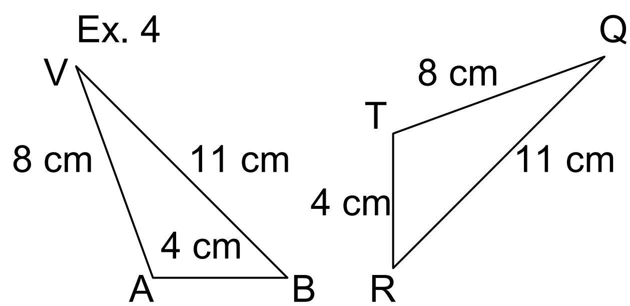
Since $\overline{AC} \cong \overline{AC}$, then ΔABC ≅ ΔADC by the SAS property.

Ex. 3



Solution:
There is no SSA property, so we cannot conclude the triangles congruent.

Ex. 4



Solution:
Since the corresponding the sides are congruent, then ΔVAB ≅ ΔQTR by SSS property.

Objective b: Understanding similar triangles

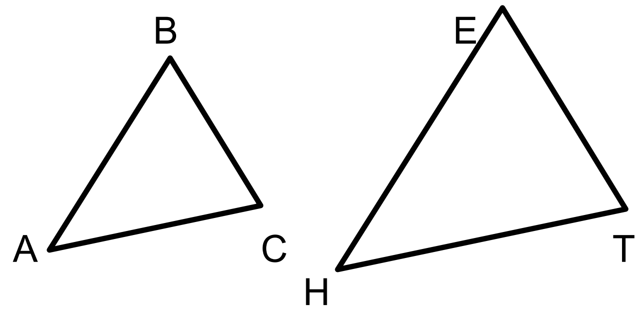
Two triangles are **similar** if they have the same shape, but not necessarily the same size. The notation for writing that triangle ABC is similar to triangle HET is **ΔABC ~ ΔHET**. The ordering of the letters shows the corresponding vertices. In this case, ∠A corresponds to ∠H, ∠B corresponds to ∠E and ∠C corresponds to ∠T.

In similar triangles, the corresponding angles are equal. Thus, if $\Delta ABC \sim \Delta HET$, then $m\angle A = m\angle H$, $m\angle B = m\angle E$, and $m\angle C = m\angle T$. The corresponding sides are not equal. However, the ratios of corresponding sides are equal since one triangle is in proportion to the other triangle.

Thus, if $\Delta ABC \sim \Delta HET$, then

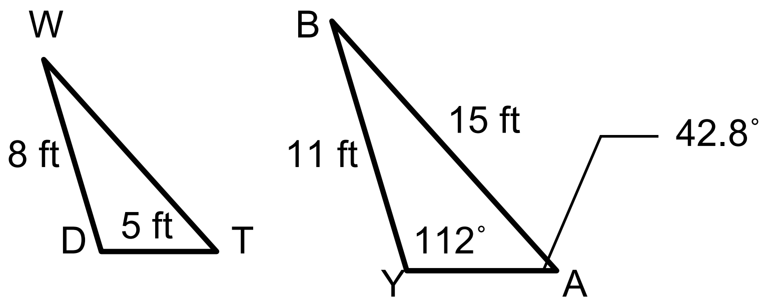
$$\frac{AB}{HE} = \frac{BC}{ET} = \frac{AC}{HT}$$

We can use proportions to then find the missing sides in a pair of similar triangles. In setting up a proportion, we always start with a pair of corresponding sides that of which we know both values. Let's try some examples.



<u>Find the missing sides and angles of the following:</u>

Ex. 5 $\Delta DWT \sim \Delta YBA$



<u>Solution:</u>

The pair of corresponding sides we will start with are DW and YB. We write the length from the smaller triangle over the length from the bigger triangle. So, our proportions are:

$\frac{DW}{BY} = \frac{WT}{BA}$ and $\frac{DW}{BY} = \frac{DT}{YA}$

$\frac{8}{11} = \frac{WT}{15}$ and $\frac{8}{11} = \frac{5}{YA}$. Now, cross multiply and solve:

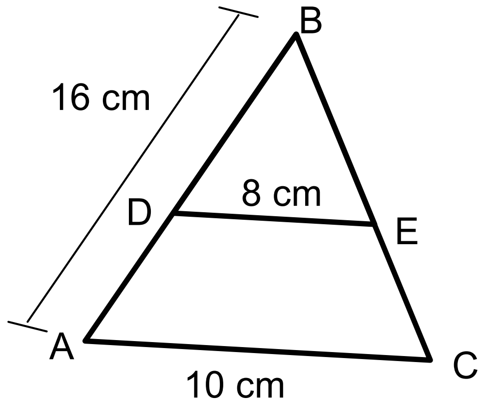
$\frac{8}{11} = \frac{WT}{15}$	$\frac{8}{11} = \frac{5}{YA}$.
$11WT = 8 \bullet 15$	$8YA = 11 \bullet 5$
$\frac{11WT}{11} = \frac{120}{11}$	$\frac{8YA}{8} = \frac{55}{8}$
$WT = 10\frac{10}{11}$ ft	$YA = 6.875$ ft.

Since $m\angle D = m\angle Y$ and $m\angle T = m\angle A$, then $m\angle D = 112^\circ$ and $m\angle T = 42.8^\circ$. Also, $m\angle W = m\angle B = 180^\circ - 112^\circ - 42.8^\circ = 25.2^\circ$.

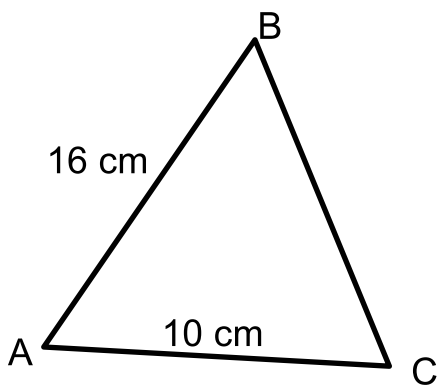
Ex. 6 On ΔABC, there is a point D on $\overline{AB}$ and a point E on $\overline{BC}$ such that $\overline{DE} \parallel \overline{AC}$. If AB = 16 cm, AC = 10 cm, and DE = 8 cm, find BD.

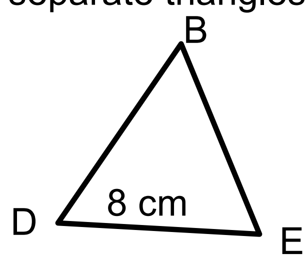
Solution:

To see how we would use similar triangles to solve this problem, let's begin by drawing a diagram:



Since $\overline{DE} \parallel \overline{AC}$, then $\overline{AB}$ is a transversal. This means that ∠ A and ∠ D are corresponding angles and therefore, m ∠ A = m ∠ D. Similarly, $\overline{BC}$ is a transversal. This implies that ∠ C and ∠ E are corresponding angles and so, m ∠ C = m ∠ E. Now, redraw the figure as two separate triangles.





Note that m ∠ B = m ∠ B. Since the measures of the corresponding angles are equal, then $\Delta ABC \sim \Delta DBE$. We know the values of AC and DE and we also have the value for AB. Since DB is the missing side and the triangles are similar, we can set-up a proportion to solve for DB.

$\frac{AC}{DE} = \frac{AB}{DB}$ (Substitute)

$\frac{10}{8} = \frac{16}{DB}$ (Cross Multiply)

$10\ DB = 8 \cdot 16$

$10\ DB = 128$ (Divide by 10)

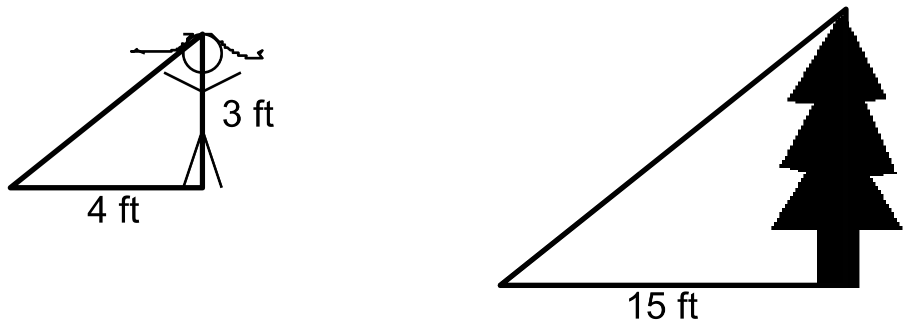
$DB = 12.8$ cm

Suppose the question had asked for AD instead of DB. In that case, we would work the problem exactly the same way and still find DB. Since AD = AB – DB, then AD = 16 cm – 12.8 cm = 3.2 cm.

Ex. 7 A tree casts a 15-foot shadow at the same time a child 3 feet tall casts a 4-foot shadow. How tall is the tree?

<u>Solution:</u>

We begin by drawing a picture:



These two triangles are similar triangles so we will take the ratio the lengths of the shadows and set it equal to the ratio of the heights:

$$\frac{\text{Length of the child's shadow}}{\text{Length of the tree's shadow}} = \frac{\text{Child's height}}{\text{Tree's height}}$$

$\frac{4}{15} = \frac{3}{h}$ Cross multiply and solve:

$4h = 15 \bullet 3$

$\frac{4h}{4} = \frac{45}{4}$

$h = 11.25$ ft

The tree is 11.25 feet tall.

Objective c: Review square roots

The square of a whole number is called a **perfect square**. So, 1, 4, 9, 16, 25 are perfect squares since $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, and $25 = 5^2$. We use the idea of perfect squares to simplify square roots. The square root of a number a asks what number times itself is equal to a. For example, the square root of 25 is 5 since 5 times 5 is 25.

The **square root** of a number a, denoted $\sqrt{a}$, is a number whose square is a. So, $\sqrt{25} = 5$.

Simplify the following (round to the nearest thousandth):

Ex. 8a $\sqrt{49}$ Ex. 8b $\sqrt{144}$

Ex. 8c $\sqrt{0}$ Ex. 8d $\sqrt{\frac{625}{64}}$

Ex. 8e $\sqrt{15}$ Ex. 8f $3\sqrt{7}$

Solution:

a) $\sqrt{49} = 7$.

b) $\sqrt{144} = 12$.

c) $\sqrt{0} = 0$.

d) $\sqrt{\frac{625}{64}} = \frac{25}{8}$.

e) Use your calculator: $\sqrt{15} = 3.8729833... \approx 3.873$.

f) Use your calculator. First, find $\sqrt{7}$ and then multiply by 3:
$3\sqrt{7} = 3(2.64575...) = 7.93725... \approx 7.937$.

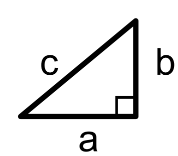
Objective d: Right Triangles and the Pythagorean Theorem

In a right triangle, there is a special relationship between the length of the legs (a and b) and the hypotenuse (c). This is known as the Pythagorean Theorem:

Pythagorean Theorem

In a right triangle, the square of the hypotenuse (c^2) is equal to the sum of the squares of the legs ($a^2 + b^2$)

$c^2 = a^2 + b^2$



Keep in mind that the hypotenuse is the longest side of the right triangle.

Determine is the following triangle is a right triangle:

Ex. 9 A triangle with sides of 7 ft, 24 ft, and 25 ft.

Solution:

Plug the values in the Pythagorean Theorem and see if you get a true statement. Since c is the longest side, then c = 25:

$c^2 = a^2 + b^2$

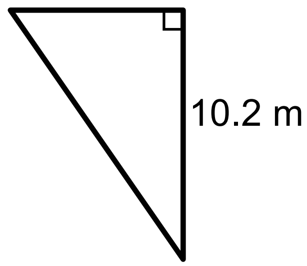
$(25)^2 = (7)^2 + (24)^2$

$625 = 49 + 576$

$625 = 625$, yes the triangle is a right triangle.

Find the length of the missing sides (to the nearest hundredth):

Ex. 10



Solution:

In this problem, we have the two legs of the triangle and we are looking for the hypotenuse:

$c^2 = a^2 + b^2$

$c^2 = (7.8)^2 + (10.2)^2$

$c^2 = 60.84 + 104.04$

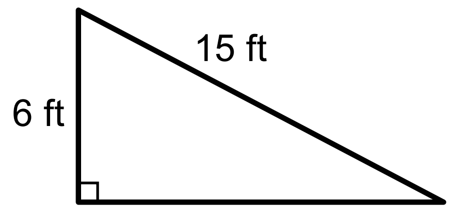
$c^2 = 164.88$

To find c, take the square root* of 164.88:

$c = \pm\sqrt{164.88} = \pm 12.8405...$

$c \approx 12.84$ m

Ex. 11



Solution:

In this problem, we have one leg and the hypotenuse and we are looking for the other leg:

$c^2 = a^2 + b^2$

$(15)^2 = (6)^2 + b^2$

$225 = 36 + b^2$ (solve for b^2)

$-36 = -36$

$189 = b^2$

To find b, take the square root of 189:

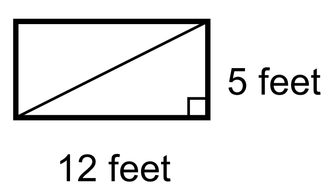
$b = \pm\sqrt{189} = \pm 13.7477...$

$b \approx 13.75$ ft

* - The equation $c^2 = 164.88$ actually has two solutions ≈ 12.84 and ≈ – 12.84, but the lengths of triangles are positive so we ignore the negative solution.

Find the length of the diagonal of the following rectangle:

Ex. 12 A rectangle that is 12 feet by 5 feet.



Solution:

First, draw a picture. Since the angles of a rectangle are right angles, then the two sides are the legs of a right triangle while the diagonal is the hypotenuse of the right triangle. Using the Pythagorean Theorem, we get:

$c^2 = a^2 + b^2$

$c^2 = (12)^2 + (5)^2$

$c^2 = 144 + 25 = 169$

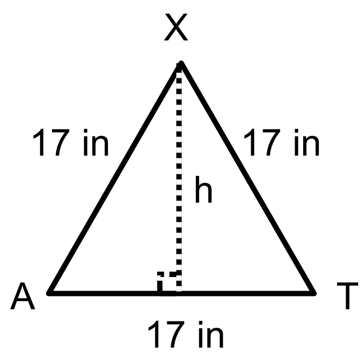
$c = \pm\sqrt{169} = \pm 13$. So, the diagonal is 13 feet.

Solve the following. Round to the nearest thousandth.

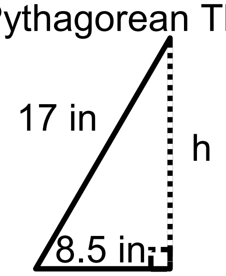
Ex. 13 Find the height of ΔAXT if each side has a length of 17 inches.

Solution:

To see how we would use the Pythagorean Theorem to solve this problem, let's begin by drawing a diagram:



The altitude (height) of the triangle forms a right angle with the base of the triangle and divides the base into two equal parts. Thus, the distance from point A and this point of intersection is 17 in ÷ 2 = 8.5 in. The right half of the triangle together with the altitude form a a right triangle with 17 in. being the length of the hypotenuse, 8.5 in. being the length of one of the legs, and the height of the equilateral triangle being the length of the other leg. Using the Pythagorean Theorem, we can find the height of the triangle:



$17^2 = (8.5)^2 + h^2$

$289 = 72.25 + h^2$ (– 72.5 from both sides)

$216.75 = h^2$ (solve for h)

$h = \pm\sqrt{216.75} = \pm 14.7224...$

So, the height is about 14.722 inches.