Section 6.5 – Graphs of the Tangent, Cotangent, Cosecant, and Secant Functions

In this section, we will look at the graphs of the other four trigonometric functions. We will start by examining the tangent function. Recall that the period of the tangent function is π. Since \( \tan(x) = \frac{\sin(x)}{\cos(x)} \), then the tangent function is undefined when \( \cos(x) = 0 \) which is when \( x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \). So, we will first sketch the graph for values of \( x \) between \( -\frac{\pi}{2} \) and \( \frac{\pi}{2} \) and then use the fact that the graph is periodic to draw the rest of the graph. We will need to look at the behavior of the function near \( -\frac{\pi}{2} \) and \( \frac{\pi}{2} \) and make a table of values to graph the tangent function:

As \( x \to -\frac{\pi}{2} \) in the fourth quadrant, \( \cos(x) \to \) a very small positive number going to zero and \( \sin(x) \to -1 \). Thus, \( \tan(x) = \frac{\sin(x)}{\cos(x)} \approx \frac{-1}{\text{small}+\#} = -\infty \).

Likewise, As \( x \to \frac{\pi}{2} \) in the first quadrant, \( \cos(x) \to \) a very small positive number going to zero and \( \sin(x) \to 1 \). Thus, \( \tan(x) = \frac{\sin(x)}{\cos(x)} \approx \frac{1}{\text{small}+\#} = \infty \).

Thus, the tangent function has vertical asymptotes at \( x = -\frac{\pi}{2} \) and \( x = \frac{\pi}{2} \).

Now, let's make a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -\frac{\pi}{2} )</th>
<th>( -\frac{\pi}{3} )</th>
<th>( -\frac{\pi}{4} )</th>
<th>( -\frac{\pi}{6} )</th>
<th>0</th>
<th>( \frac{\pi}{6} )</th>
<th>( \frac{\pi}{4} )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{\pi}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>tan(x)</td>
<td>undef.</td>
<td>(-\sqrt{3})</td>
<td>(-1)</td>
<td>(-\frac{\sqrt{3}}{3})</td>
<td>0</td>
<td>(\frac{\sqrt{3}}{3})</td>
<td>1</td>
<td>(\sqrt{3})</td>
<td>undef.</td>
</tr>
</tbody>
</table>

Now, will put the information on a graph:
Now, draw a smooth curve:

This is the graph of $y = \tan(x)$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Since it is periodic, this shape repeats and thus we get the following graph:
\[ y = \tan(x) \]

**Properties of the Tangent Function**

1) The domain is \( \{x | x \neq \frac{(2n+1)\pi}{2} \} \) where \( n \) is an integer).

   The range is \( (-\infty, \infty) \).

2) The tangent function is odd so it is symmetric with respect to the origin.

3) The tangent function is periodic with a period of \( \pi \).

4) The \( x \)-intercepts are \( \{(n\pi, 0) | n \text{ is an integer}\} \). The \( y \)-intercept is \( (0, 0) \).

5) The vertical asymptotes are \( \{ x = a | a = \frac{(2n+1)\pi}{2} \} \) where \( n \) is an integer).

We will now examine the cotangent function. Recall that the period of the cotangent function is \( \pi \). Since \( \cot(x) = \frac{\cos(x)}{\sin(x)} \), then the cotangent function is undefined when \( \sin(x) = 0 \) which is when \( x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots \). So, we will first sketch the graph for values of \( x \) between 0 and \( \pi \) and then use the fact that the graph is periodic to draw the rest of the graph. We will need to look at the behavior of the function near 0 and \( \pi \) and make a table of values to graph the cotangent function:

As \( x \to 0 \) in the first quadrant, \( \sin(x) \to 0 \) a very small positive number going to zero and \( \cos(x) \to 1 \). Thus, \( \cot(x) = \frac{\cos(x)}{\sin(x)} \approx \frac{1}{\text{small}^+} = \infty \). Likewise, As \( x \to \pi \) in the second quadrant, \( \sin(x) \to 0 \) a very small positive number going to zero and \( \cos(x) \to -1 \). Thus, \( \cot(x) = \frac{\cos(x)}{\sin(x)} \approx \frac{-1}{\text{small}^+} = -\infty \). Thus, the cotangent function has vertical asymptotes at \( x = 0 \) and \( x = \pi \).

Now, let's make a table of values.
<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{2\pi}{3}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\frac{5\pi}{6}$</th>
<th>$\frac{\pi}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cot(x)</td>
<td>undef.</td>
<td>$\sqrt{3}$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>0</td>
<td>$-\frac{\sqrt{3}}{3}$</td>
<td>$-1$</td>
<td>$-\sqrt{3}$</td>
<td>undef.</td>
</tr>
</tbody>
</table>

Now, will put the information on a graph:

![Graph of cot(x)](image)

Now, draw a smooth curve:

![Smooth curve of cot(x)](image)

This is the graph of $y = \cot(x)$ on the interval $[0, \pi]$. Since it is periodic, this shape repeats and thus we get the following graph:

$$y = \cot(x)$$
Properties of the Cotangent Function

1) The domain is \( \{ x \mid x \neq n\pi \text{ where } n \text{ is an integer} \} \).
   The range is \( (-\infty, \infty) \).
2) The cotangent function is odd so it is symmetric with respect to the origin.
3) The cotangent function is periodic with a period of \( \pi \).
4) The x-intercepts are \( \{(\frac{2n+1}{2}\pi, 0) \mid n \text{ is an integer}\} \).
   There is no y-intercept.
5) The vertical asymptotes are \( \{ x = a \mid a = n\pi \text{ where } n \text{ is an integer}\} \).

Objective 1: Graph Functions of the Form \( y = \tan(\omega x) \) and \( y = \cot(\omega x) \).

Just with the sine and cosine function, the amplitude \(|A|\) is a vertical stretch/compression and \( \frac{1}{\omega} \) is a horizontal stretch/compression. The main difference is the period. Since the tangent and cotangent functions have a period of \( \pi \), a tangent or cotangent function with a horizontal stretch/compression of \( \frac{1}{\omega} \) will have a period of \( T = \frac{\pi}{\omega} \).

Use transformations to sketch the graph of the following:

Ex. 1a \( y = -\frac{1}{4}\tan(3x) + 2 \) \hspace{1cm} Ex. 1b \( y = 2\cot\left(\frac{x}{4}\right) - 1 \)

Solution:
a) The amplitude is $|\frac{-1}{4}| = \frac{1}{4}$ and the period is $T = \frac{\pi}{3}$. The graph is also reflected across the x-axis, vertically compressed by a factor of $\frac{1}{4}$, and shifted up 2 units. Since the period is $1/3$ the size of the period for $\tan(x)$, instead of going from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we will graph from $-\frac{\pi}{2} \div 3 = -\frac{\pi}{6}$ to $\frac{\pi}{2} \div 3 = \frac{\pi}{6}$.

b) The amplitude is $|2| = 2$ and the period is $T = \frac{\pi}{\frac{\pi}{4}} = 4$. The graph is not reflected across the x-axis but is vertically stretched by a factor of 2, and shifted down 1 unit.
The next function we want to explore is the cosecant function. The period for the cosecant function is $2\pi$. Since $\csc(x) = \frac{1}{\sin(x)}$ then the cosecant function is undefined when $\sin(x) = 0$ which is when $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$. These will act as the vertical asymptotes. To obtain the graph of the cosecant, we can first draw the sine function and then use the reciprocal relationship to "turn the graph inside out:"

Properties of the Cosecant Function:
1) The domain is $\{x| x \neq n\pi \text{ where } n \text{ is an integer}\}$. The range is $(\infty, -1] \cup [1, \infty)$.
2) The cosecant function is odd so it is symmetric with respect to the origin.
3) The cosecant function is periodic with a period of $2\pi$.
4) There are no intercepts.
5) The vertical asymptotes are $\{ x = a | a = n\pi \text{ where } n \text{ is an integer}\}$. 
The final function we want to explore is the secant function. The period for the secant function is $2\pi$. Since $\sec(x) = \frac{1}{\cos(x)}$ then the cosecant function is undefined when $\sin(x) = 0$ which is when $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

These will act as the vertical asymptotes. To obtain the graph of the secant, we can first draw the cosine function and then use the reciprocal relationship to "turn the graph inside out:"

![Graph of the secant function]

Now, erase the cosine function:

![Erasd Graph of the secant function]

**Properties of the Secant Function:**

1) The domain is $\{x| x \neq \frac{(2n+1)\pi}{2}\}$ where $n$ is an integer.

   The range is $(-\infty, -1] \cup [1, \infty)$.

2) The secant function is even so it is symmetric with respect to the y-axis.

3) The secant function is periodic with a period of $2\pi$.

4) There are no x-intercepts. The y-intercept is $(0, 1)$.

5) The vertical asymptotes are $\{ x = a | a = \frac{(2n+1)\pi}{2} \}$ where $n$ is an integer.
Objective 2: Graph Functions of the Form \( y = A \csc(\omega x) \) and \( y = A \sec(\omega x) \).

Just with the sine and cosine function, the amplitude \(|A|\) is a vertical stretch/compression and \( \frac{1}{\omega} \) is a horizontal stretch/compression. Since the cosecant and secant functions have a period of \( 2\pi \), a cosecant or secant function with a horizontal stretch/compression of \( \frac{1}{\omega} \) will have a period of \( T = \frac{2\pi}{\omega} \).

**Use transformations to sketch the graph of the following:**
Ex. 2a \( y = -\csc(\pi x) - 2 \)  
Ex. 2b \( y = 2\sec\left(\frac{\pi}{3}x\right) + 5 \)

**Solution:**

a) The amplitude is \(|-1| = 1\). The period is \( T = \frac{2\pi}{\pi} = 2 \). The function is reflected across the x-axis, compressed by a factor of \( \frac{1}{\pi} \) and shifted down 2 units.

![Graph of y = -csc(\pi x) - 2]

b) The amplitude is \(|2| = 2\). The period is \( T = \frac{2\pi}{\frac{\pi}{3}} = 6 \). The function is reflected across the x-axis, compressed by a factor of \( \frac{3}{\pi} \) and shifted up 5 units.
Sometimes it might be easier to first draw the sine or cosine function first. We would draw it reflected, stretched/compressed, and shifted in the same fashion as the corresponding cosecant or secant function and then turn the graph inside out. So, in part b, we could have drawn the corresponding cosine function and use that to obtain the secant graph: